Hello CS103!!

- What have we learned?
- A recap on proofs
- Practice exam #1

  1. Proof writing
  2. Ternary conditionals
  4. Set theory

We won't cover #3... but that doesn't mean you shouldn't do it!!

What have we learned? (serious question)

Mathematical Foundations of computer science $\approx$ A **toolkit** for reasoning about computer science

This is a BIG DEAL!

Is it obvious that the "best" way to represent the world is 0s and 1s?

not to me... but y'all are really smart

Why are some problems easy to solve? Why are others hard?

time, memory, randomness... true, false, search...

Time dilation, 3blue1brown, puzzles...

fun stuff? maybe?

Topics

- Proofs
- Sets

$\Big\}$ Benson

- Propositional logic
- First order logic

$\Big\}$ Stanley

Quick plug: Guide to Negations

TBH I still use this :)

A recap on proofs

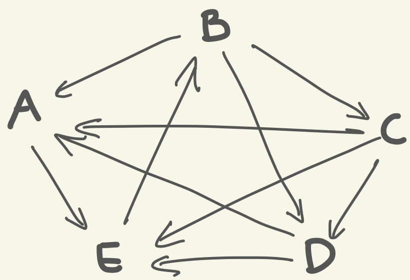
Where do we start? Direct, indirect? Contradiction, contrapositive?

Generally, we don't know. BUT problems can guide us!. Not a ton we can do.

Example 1: Tournament champions

Problem set 2 9

Remember tournaments?



Player B has beaten A, D, C and lost to E.

ii. Let T be an arbitrary tournament and c be a player in that tournament. Prove the following statement: if c won more games than anyone else in T or is tied for winning the greatest number of games, then c is a tournament champion in T.

... Where should we start with this problem?

Take a second. Do you remember the hint? (Negate the definition of tournament champion.)

ii. Let T be an arbitrary tournament and c be a player in that tournament. Prove the following statement: if c won more games than anyone else in T or is tied for winning the greatest number of games, then c is a tournament champion in T.

→ vague, unhelpful. → a tournament has multiple people in it... so we ought to talk about multiple.

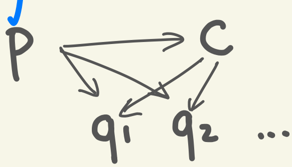
We love definitions!

FTSOC assume c is not a tournament champion. Then there is a player p such that

1. c lost to p
2. If c beat q, then p also beat q.

Not so easy to obtain. That's why we practice negations.

Now we can get a nice picture.



Punchline: Definitions are a way to get free information! That's our "Assumptions" column.

Example 2: Strictly increasing functions

Problem set 3.3

A function $f : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing if the following is true:

$$\forall x \in \mathbb{Z}. \forall y \in \mathbb{Z}. (x < y \longrightarrow f(x) < f(y))$$

Now, let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be strictly increasing functions. Prove that $g \circ f$ is strictly increasing.

What do we want to show?

From the definition:

For any arbitrary $x, y \in \mathbb{Z}$, if $x < y$, then $g \circ f(x) < g \circ f(y)$.

Great! Now we have x, y... Let's try to get $g \circ f(x)$.

$$x < y \longrightarrow f(x) < f(y)$$ f is strictly increasing

$$\longrightarrow g \circ f(x) < g \circ f(y)$$ and so is g

Punchline: Definitions guide us!

That's our "want to show" column.

Example 3. Set theory

Practice exam 1.4

Let's say we have sets A and B.
How do we show $A \subseteq B$?

$$\boxed{\text{Pick } x \in A. \text{ We will show that } x \in B.}$$

Great! We all solemnly swear that, if asked to prove $A \subseteq B$, this is THE FIRST THING we will try.

Also, to show $A = B$, prove

1) $A \subseteq B$ and 2) $B \subseteq A$

Example 3. Set theory (for real)

Let A and B be sets where $A \subseteq B$. Below are two statements about A and B. For each statement, determine if it is

- Always True,
- Never True, or
- Sometimes True (depends on choice of A and B).

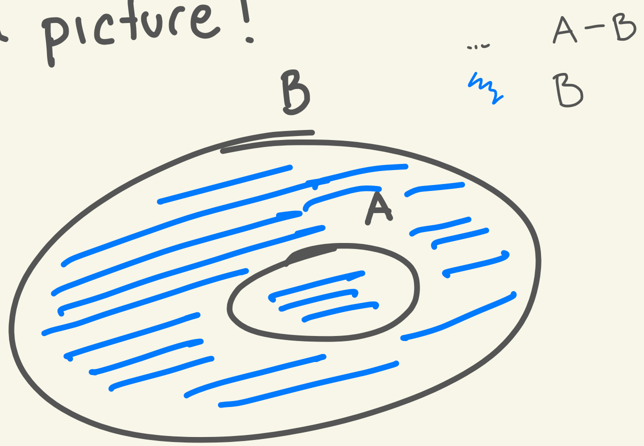
If the statement is always true or never true, provide a formal proof. If the statement is sometimes true, provide one example of sets A and B where the statement is true and one example of sets A and B where the statement is false. (Remember that, in your examples, you need to have $A \subseteq B$.)

i. ***Statement 1:*** $(A - B) \cup B = A$.

ii. ***Statement 2:*** $(B - A) \cup A = B$.

Statement 1: $(A - B) \cup B = A$

- Draw a picture!



Doesn't look true..

Statement 1: $(A-B) \cup B = A$

- Play with examples

Our favorite example?

$A = \{\}$, $B = \{\}$

$(A-B) = \{\}$

So $\{A-B\} \cup B = \{\} = A$ ✓

$A = \{\}$, $B = \{1\}$

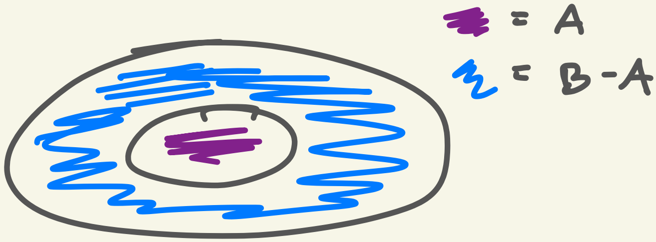
$(A-B) = \{\}$

$(A-B) \cup B = \{1\} \neq \{\}$ ✗

→ Sometimes true.

Statement 2: $(B-A) \cup A = B$

- Draw



Looks good!
(You can work out examples on your own.)

Proof time!!

- Let A, B be arbitrary sets such that $A \subseteq B$.
- We want to show that $(B-A) \cup A = B$.
- To do so, we prove $(B-A) \cup A \subseteq B$ and $B \subseteq (B-A) \cup A$.

$(B - A) \cup A \subseteq B$. Pick an arbitrary $x \in (B - A) \cup A$. We will show that $x \in B$.

There is a natural way to make this easier!!
$\longrightarrow$ Cases.

Case 1: $x \in B - A$. Then $x \in B$, as required.

Case 2: $x \in A$. $A \subseteq B$, so $x \in B$.

Now, we show that $B \subseteq (B - A) \cup A$. Pick an arbitrary $x \in B$. We will show that $x \in (B - A) \cup A$.

Is there a way to simplify this?

Case 1: $x \in A$. Then $x \in (B - A) \cup A$.

Case 2: $x \notin A$. Because $x \in B$, then $x \in B - A$.

And so we're done!

Probably won't use this

Example ~~3~~ 4. Adjacent Pythagorean Triples

A Pythagorean Triple is a triple $a, b, c \in \mathbb{N}$ where $a^2+b^2=c^2$. Prove that if (a,b,c) is a triple, then $(a+1, b+1, c+1)$ is not a Pythagorean Triple.

Where to start?

$$\underbrace{(a+1)^2+(b+1)^2}_{\text{what we know (this is just math).}} = a^2+2a+1+b^2+2b+1$$

$$= a^2+b^2+2a+2b+2$$

$$= c^2+2a+2b+2$$

$$(c+1)^2 = c^2+2c+1$$

... is $2c+1$ equal to $2a+2b+2$?

NO — parity!!

NOTE that we did NOT start with $(a+1)^2+(b+1)^2=(c+1)^2$.

However, if we would like to, we can say: For the sake of contradiction, assume that $(a+1)^2+(b+1)^2=(c+1)^2$